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# On response of nonlinear oscillators with random frequency of excitation

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**Abstract**

Some types of nonlinear oscillators, for which the frequency of excitation is stochastic, are investigated. The paper consists of two parts. In the first part equations of motion are linearized. With the aid of stochastic averaging differential equations for the mean and variance of the process are obtained and integrated numerically. This approach is applicable for weakly nonlinear oscillators.

The case of strong nonlinearity is considered in the second part. Making use of computer simulation, a number of stochastic realizations of the process are computed. The stochastic process is characterized by the mean and standard deviation of these realizations. Calculations have been carried out for the Duffing, Ueda and van der Pol equations and for forced vibrations of a pendulum. These calculations show that if attractors exist then the deterministic vibrations (which may be chaotic) turn regular by adding noise and the motion terminates in a stable fixed point or on a limit cycle.


## 1. Introduction

Nonlinear stochastic vibrations are investigated in many textbooks and papers (see e.g. [1–3]). New approaches and methods of solution are often tested on well-known oscillators. For this purpose, the Duffing model is used in several papers, such as Refs. [4–15].

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The Duffing equation has the form

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx + rx^3 = s\cos\omega t + f(t). \tag{1}$$

In most studies $p$, $q$, $r$, $s$, $\omega$ are regarded as constants. For vibrating systems the dissipation coefficient $p$ must be positive. If $q>0$, $r>0$ an oscillator with hardening stiffness is obtained, this case is discussed in Refs. [4,5,7,10,12,13,15]. The other case, where $q$ and $r$ have opposite signs, is analysed in Refs. [6,8,11]. In Refs. [9,16] the variable $q$ is taken in the form

$$q = q_0[1 + \gamma\cos\omega t + \delta\sin\omega t], \tag{2}$$

where $q_0$, $\gamma$, $\delta$ are constants.

Different ways of introducing noise into the nonlinear system are distinguished below:

(i) *Additional noise*: Here it is assumed that $f(t) = \alpha\xi(t)$. In most papers $\xi$ is a zero mean Gaussian noise. In Ref. [14] Poisson noise excitation is assumed.

(ii) *Multiplicative noise*: The coefficients in Eq. (1) are random.

(iii) *Bounded noise*: In this case the frequency and the phase of excitation are random [4,5,11,15]. In Ref. [11] the frequency term is taken in the form $f(t) = \mu\sin(\Omega t + \psi)$, $\psi = \sigma B(t) + \Gamma$, where $\mu$, $\Omega$, $\sigma$ are positive constants, $B(t)$ is a unit Wiener process and $\Gamma$ a random variable. In Ref. [15] a similar approach can be found.

In Refs. [4,5] the noise is modelled as the solution of the equation

$$\frac{d^2X}{dt^2} + \beta\frac{dX}{dt} + v^2X = v\sqrt{\beta}W(t), \tag{3}$$

where $\beta$, $v$ are deterministic constants and $W(t)$ means Gaussian white noise.

Some papers are concerned with other nonlinear equations, such as the stochastic van der Pol equation investigated in Refs. [14,17]; the forced motion of a pendulum is discussed in Refs. [18,19].

Different methods of solution have been applied. According to the conventional approach [4–6,12,14,17,18] Eq. (1) is linearized with the aid of equivalent linearization techniques. The stochastic averaging method is a powerful approximate technique for the prediction of nonlinear oscillator response. In Refs. [6,11,16] the Melnikov method for calculating the homoclinic threshold is applied. The pseudoforce theory is developed in Ref. [7]. In Ref. [9] the method of multiple scales is used. A wavelet-based solution is proposed in Ref. [10].

Computer simulation is used in several papers since the methods for solving stochastic nonlinear equations are in most cases rather complicated. For this purpose the Euler scheme [8] or the fourth-order Runge–Kutta method [4,5,10,13,16] is applied. In some papers the Monte Carlo simulation is used as well.

The relationship between the chaotic and stochastic motion is quite interesting. It is well known that the stochastic excitation may bring the regular motion into the chaotic state (stochastic chaos). Here it is appropriate to cite Szemplińska–Stupnicka [20], who raised the question: Can chaotic motion be interpreted as nonstationary free vibration with randomly modulated amplitude and phase? On the contrary, there are some papers in which it is demonstrated that the noise effect may stabilize the system [8,15] and too strong a noise destroys the signal [17]. The transition from deterministic responses to purely random results is discussed in Ref. [6]. From here it follows that the relation between deterministic chaos and stochastic vibrations, if any, is not well understood.

The excitation frequency $\omega$ is for physical reasons not strictly a constant, but carries some small fluctuations. For this reason $\omega$ can be considered a narrow-band random variable. This case for the Duffing attractor was discussed by Lepik [21]. It turned out that by adding noise to the frequency $\omega$ the initially chaotic motion becomes regular and is terminated in one of the focuses. The authors think that is an interesting result. Unfortunately they have not found any other papers on this topic. With the view of providing some physical insights into this problem the present paper is dedicated to the analysis of nonlinear equations with random frequency of excitation.

The paper consists of two different parts. In Section 3 the weakly nonlinear equations are considered. In the second part (Sections 4–7) the computer simulation method is applied for analysing the response of some strongly nonlinear oscillators.

## 2. Problem statement

Consider nonlinear differential equation

$$\ddot{x} + g(t, x, \dot{x}) = s \cos \omega t, \quad 0 \leqslant t \leqslant T \tag{4}$$

with the boundary conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$. Dots stand for time derivatives, $g$ is a prescribed deterministic function, $s$, $x_0$, $\dot{x}_0$ are deterministic constants. The quantity $\omega$ has the form

$$\omega = \omega_0[1 + \alpha\xi(t)], \tag{5}$$

where $\omega_0$ and $0 \leqslant \alpha \leqslant 1$ are constants; $\xi(t)$ represents a Gaussian white noise with zero mean and standard deviation $\sigma = 1$. The coefficient $\alpha$ characterizes the noise intensity (for $\alpha = 0$ the motion is deterministic). The aim is to integrate Eq. (4) and explore the effect of randomness to the nonlinear vibrations.

For interpreting the achieved results, the knowledge of the fixed points of system (4) and their type is useful. The type of the fixed points is determined for the linearized and unperturbed system. In the case of a nonlinear system it is expedient to make use of the Hartman–Grobman theorem [22] according to which the fixed points of the linear system maintain their type also in the case of the corresponding nonlinear system (with the exception of the centres and degenerated points for which the character of the fixed points may change).

## 3. Solution of the linearized equation

Consider again (4); the corresponding deterministic equation has the form

$$\ddot{x}_D + g(t, x_D, \dot{x}_D) = s \cos \omega_0 t. \tag{6}$$

Next the noise-induced deviation $\delta x = x - x_D$ is introduced. In view of (1, 4)

$$\delta\ddot{x} + g(t, x, \dot{x}) - g(t, x_D, \dot{x}_D) = s(\cos \omega t - \cos \omega_0 t), \tag{7}$$

where the stochastic variable $\omega$ is defined according to Eq. (5).

To be more specific, in this section the Duffing equation $g(t, x, \dot{x}) = p\dot{x} + qx + rx^3$ is considered. Oscillators with weak nonlinearity for which $x^3 \approx x_D^3 + 3x_D^2\delta x$ are discussed. For this case (7) can be presented in the form

$$\begin{aligned} \delta\dot{x} &= \delta y, \\ \delta\dot{y} &= -p\delta y - q\delta x - 3rx_D^2\delta x + s\phi(t, \xi), \end{aligned} \tag{8}$$

where $\phi(t, \xi) = \cos \omega t - \cos \omega_0 t$.

Expanding the function $\phi(t, \xi)$ into trigonometric series, we obtain

$$\begin{aligned} \phi(t, \xi) = {} & \cos \omega_0 t[-\tfrac{1}{2}(\alpha\omega_0 t\xi)^2 + \tfrac{1}{24}(\alpha\omega_0 t\xi)^4 - \cdots] \\ & - \sin \omega_0 t[\alpha\omega_0 t\xi - \tfrac{1}{6}(\alpha\omega_0 t\xi)^3 + \cdots]. \end{aligned} \tag{9}$$

Since $\xi$ is a normally distributed random variable, it has the moments (symbol $E$ denotes the mean):

$$E(\xi) = 0, \quad E(\xi^2) = 1, \quad E(\xi^3) = 0, \quad E(\xi^4) = 3.$$

In view of these equalities the mean of the function $\phi$ is

$$E[\phi(t, \xi)] = -\tfrac{1}{2}(\alpha\omega_0 t)^2 \cos \omega_0 t[1 - \tfrac{1}{4}(\alpha\omega_0 t)^2 + \cdots]. \tag{10}$$

Stochastic averaging of Eq. (8) gives

$$\begin{aligned} E(\delta\dot{x}) &= E(\delta y), \\ E(\delta\dot{y}) &= -pE(\delta y) - (q + 3rx_D^2)E(\delta x) + sE(\phi). \end{aligned} \tag{11}$$

Introducing the second-order moments

$$M_x = E[(\delta x)^2], \quad M_{xy} = E(\delta x\delta y), \quad M_y = E[(\delta y)^2] \tag{12}$$

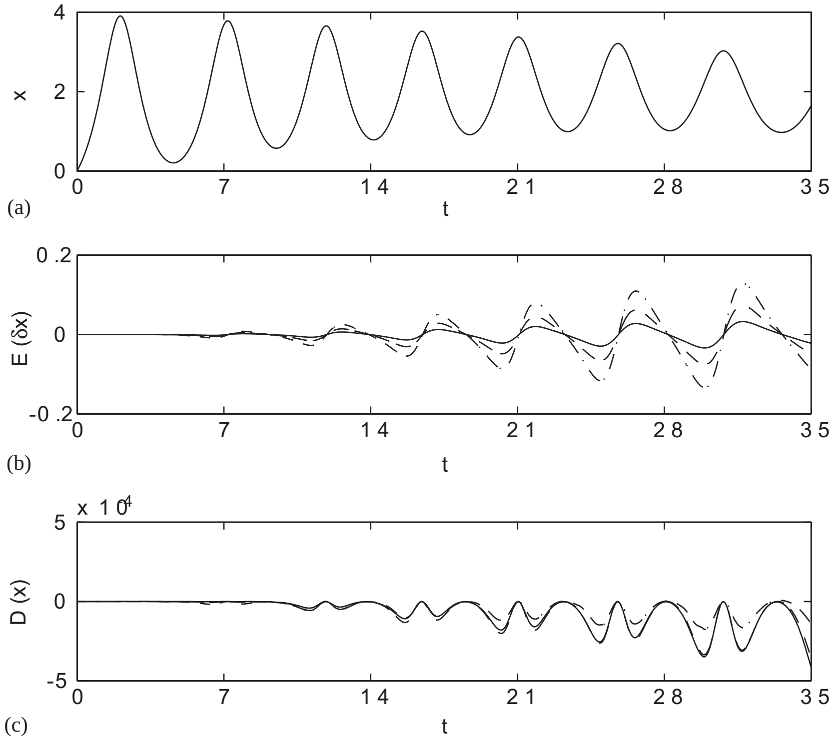


Fig. 1. Weakly nonlinear Duffing equation for $p = 0.05$, $q = -1$, $r = 0.2$, $s = 1$, $\omega_0 = 0.05$, $x_0 = 0$, $\dot{x}_0 = 1$; (a) time history of deterministic vibrations, (b) expectation of the noise-induced deviation $E(\delta x)$, (c) the variance $D(x)$: —— $\alpha = 0.1$, $---$ $\alpha = 0.15$, $- \cdot - \cdot -\alpha = 0.2$.

and taking into account Eq. (11), the following system of equations is obtained:

$$\dot{M}_x = 2M_{xy},$$

$$\dot{M}_{xy} = -(q + 3rx_D^2)M_x - pM_{xy} + M_y + sE(\delta x)E(\phi), \tag{13}$$

$$\dot{M}_y = -2(q + 3rx_D^2)M_{xy} - 2pM_y + 2sE(\delta y)E(\phi).$$

This system can be integrated according to the following algorithm.

*Step* 1: Solve Eq. (6) for boundary conditions $x_D(0) = x_0$, $y_D(0) = \dot{x}_0$.

*Step* 2: Calculate $E(\phi)$ from Eq. (10).

*Step* 3: Integrate Eq. (11) for boundary conditions $E[\delta x(0)] = E[\delta y(0)] = 0$.

*Step* 4: Integrate Eq. (13) for $M_x(0) = M_{xy}(0) = M_y(0) = 0$.

*Step* 5: Calculate

$$E(x) = x_D + E(\delta x),$$
$$D(x) = E[(x_D + \delta x - E(\delta x))^2] = E[(\delta x)^2] - [E(\delta x)]^2. \tag{14}$$

Knowing the mean $E(x)$ and variance $D(x)$ over the time interval $t \in [0, T]$ it is usually sufficient to characterize the stochastic process (4). In the case of necessity, higher moments as skewness and kurtosis can be calculated.

As an example the case $p = 0.05,\ q = -1,\ r = 0.2,\ x_0 = 0,\ \dot{x}_0 = 1$ is considered; the results are plotted in Fig. 1.

This method can be applied only in the case of weak nonlinear systems for which the higher powers of $\delta x$ can be neglected. Strong nonlinear oscillators are considered in the following sections.

## 4. Computer simulation

For numerical integration of Eq. (4) the time interval $t \in [0, T]$ is discretized so that $0 \leqslant t_1 < t_2 < \cdots < t_k \leqslant 1$; here $t_i,\ i = 1, 2, \ldots, k$ are discretizion points and $k$ is the number of these points. Making use of the Gaussian pseudorandom number generator, the variable $\xi$ is discretized in the same points; for intermediate instants the values of $\xi$ are calculated by some appropriate

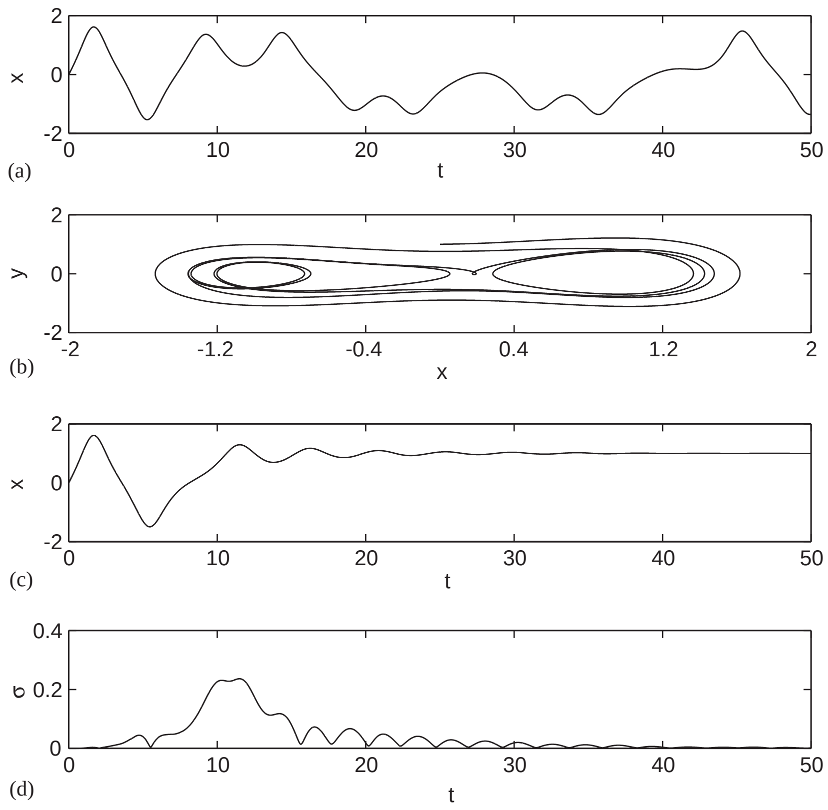


Fig. 2. Duffing equation (4) for $p = 0.25,\ q = -1,\ r = 1,\ s = 0.3,\ \omega_0 = 1,\ x_0 = 0,\ \dot{x}_0 = 1$. In Figs. 2–11 subdiagrams (a)–(d) have the following meaning: (a) time history and (b) phase diagram in the case of deterministic motion; (c) time history and (d) standard deviation for the stochastic realizations.

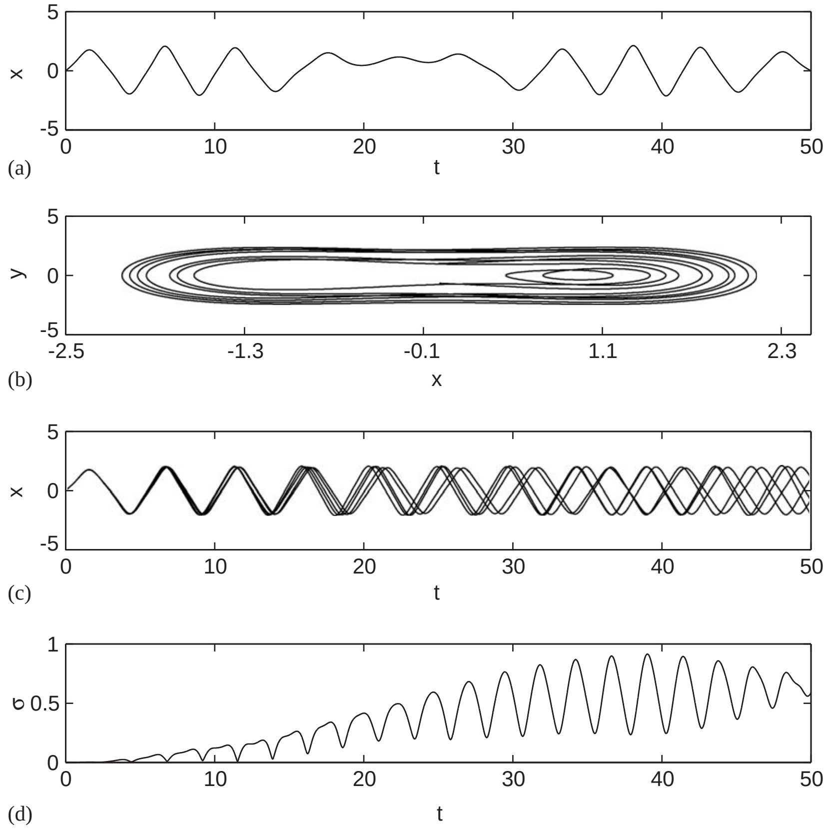


Fig. 3. Duffing equation (1) for $p = 0,\ q = -1,\ r = 1,\ s = 0.3,\ \omega_0 = 1,\ x_0 = 0,\ \dot{x}_0 = 1.$

interpolation method. Now the function $\cos \omega t$ is continuous and for integrating (4) the technique used in the case of deterministic systems can be applied. Of course, this is an approximation of the actual stochastic process for which $\xi$ is not differentiable and Itô-type equations hold. This approach is supported by the fact that in reality the forcing term $F = s \cos \omega t$ is continuous by physical reasons.

Integration of Eq. (4) is repeated for $N$ independent different sequences $\{\xi_i\}$; in this way $N$ realizations of the random process are obtained. From these data the mean, the variance and the standard deviation are calculated with the aid of the formulae

$$E[x(t)] = \frac{1}{N} \sum_{v} x^{(v)}(t),$$
$$D[x(t)] = \frac{1}{N-1} \sum_{v} [x^{(v)}(t) - E[x(t)]]^2, \quad \sigma = \sqrt{D[x(t)]}. \tag{15}$$

Here $v$ is the number of the $v$th realization.

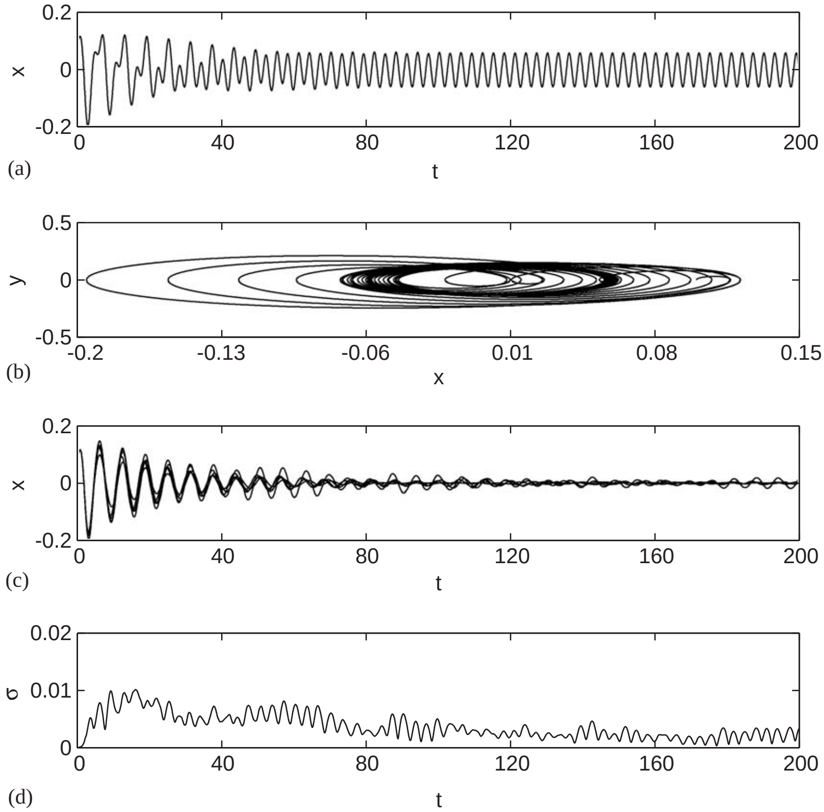


Fig. 4. Duffing equation (1) for $p = 0.09,\ q = 1,\ r = 0.3,\ s = 0.2,\ \omega_0 = 2.1,\ x_0 = 0.1,\ \dot{x}_0 = 0$.

According to this scheme, computer simulations were carried out for a number of problems. The fourth-order Runge–Kutta method with the adapted stepsize was used. It turned out that already a small number of realizations ($N < 10$) enables estimation of various statistical features of the solution.

Some results for $\alpha = 0.2$ are plotted in Figs. 2–10. To preserve clarity of these plots for $N$, a small number $N = 5$ was taken. Each plot in Figs. 2–10 consists of four parts. In parts (a) and (b) the time history and phase diagram for deterministic motion $\alpha = 0$ are plotted. In part (c) stochastic realizations are presented; in part (d) the standard deviation as a time function is shown.

## 5. Duffing oscillator

For this oscillator the function $g$ in Eq. (4) has the form

$$g(t, x, \dot{x}) = p\dot{x} + qx + rx^3. \tag{16}$$

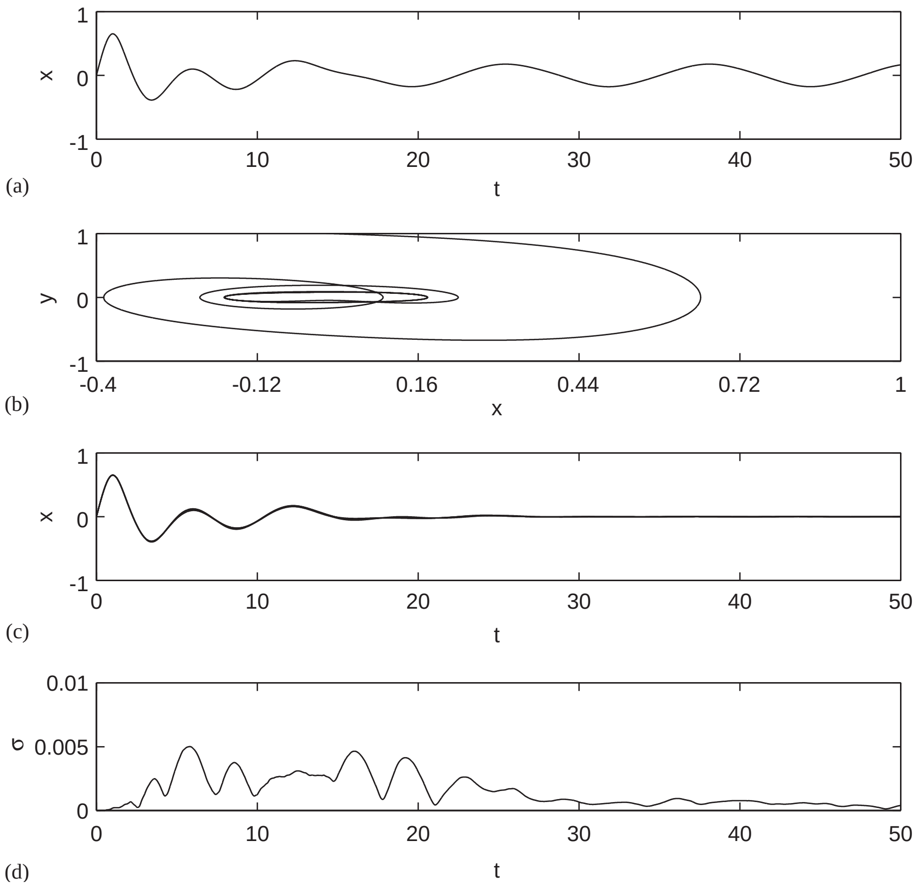


Fig. 5. Duffing equation (1) for $p = 0.4,\ q = 1,\ r = 4,\ s = 0.15,\ \omega_0 = 0.5,\ x_0 = 0,\ \dot{x}_0 = 1$.

The unforced equation $s = 0$ has three fixed points $\bar{x}_1 = \bar{y}_1 = 0$ and $\bar{x}_{2,3} = \pm\sqrt{-q/r}$, $\bar{y}_{2,3} = 0$ (the notation $y = \dot{x}$ is introduced). The eigenvalues of these fixed points are [22]

$$\lambda = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q - 3r\bar{x}_i^2} \quad (i = 1, 2, 3). \tag{17}$$

Next some special cases are considered.

(i) The oscillator with softening stiffness $p>0,\ q<0,\ r>0$. In the case of the fixed point $\bar{x}_1 = 0$ it follows from Eq. (17) that $\lambda_1<0,\ \lambda_2>0$ and this is a saddle point. As to $\bar{x}_{2,3}$ then $\lambda_1<0,\ \lambda_2<0$; if $p^2 + 8q>0$ these are stable modes, in the opposite case $p^2 + 8q<0$ the eigenvalues are complex numbers and the fixed points are stable focuses. So for this type oscillator always two stable fixed points exist (two-well oscillator).

Computer simulation results for a typical case are presented in Fig. 2. Deterministic motion is chaotic, stable focuses are at $\bar{x} = \pm 1$. By adding noise with $\alpha = 0.2$ the motion turns regular and terminates in the focus $x = 1$. The standard deviation $\sigma$ is maximal around $t \approx 10$ and with increasing time approaches to zero.

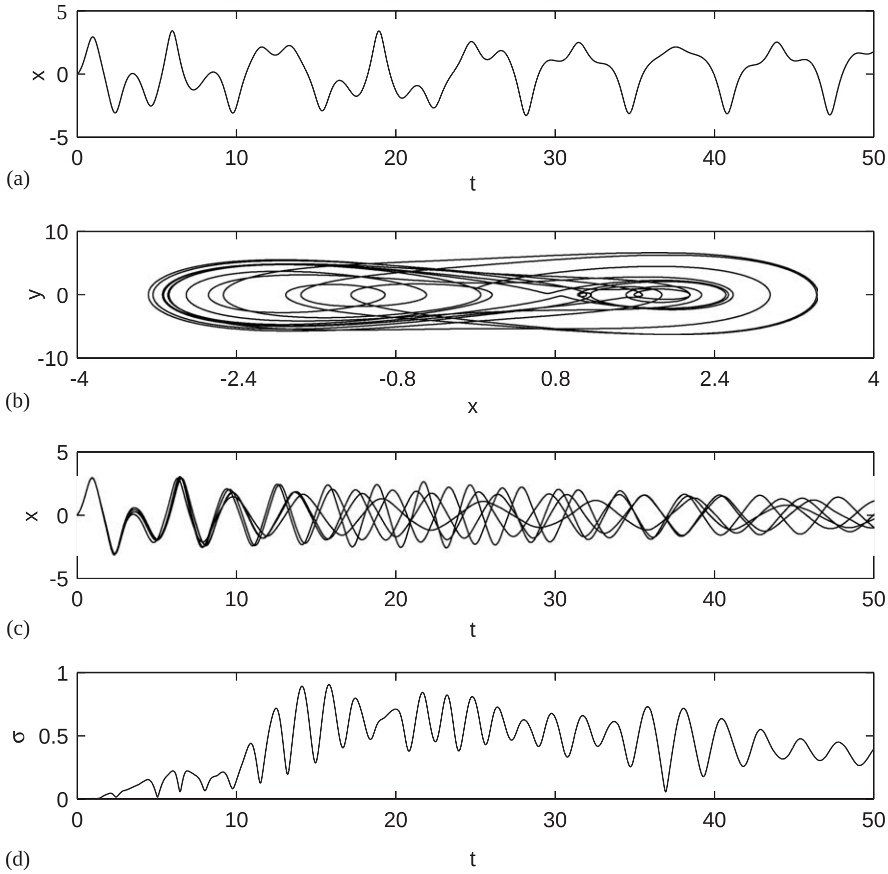


Fig. 6. Ueda oscillator; Eq. (1) for $p = 0.05,\ q = 0,\ r = 1,\ s = 7.5,\ \omega_0 = 1,\ x_0 = 0,\ \dot{x}_0 = 1$.

Calculations with other parameter values indicated that the situation, where some of the stochastic realizations are attracted by the focus $\bar{x} = 1$ and other—by the other focus $\bar{x} = -1$, may exist.

Of interest also is the case $p = 0$ (attractor without dissipation). The fixed point $\bar{x}_1 = 0$ is again a saddle point. As to the points $\bar{x}_{2,3}$ then $\lambda_{1,2} = \pm\sqrt{2q}$; since $q<0$ both eigenvalues are imaginary and these points are centres. It follows from here that in the case $p = 0$ no attractor exists.

Computer simulation results for $p = 0,\ q = -1,\ r = 1$ are plotted in Fig. 3. It follows from this figure that all stochastic realizations are different and do not converge to a unit solution. The standard deviation $\sigma$ has an increasing tendency in time.

(ii) In the case of a hard type attractor $p>0,\ q>0,\ r>0$ and only one fixed point $\bar{x}_1 = 0$ exists. It follows from Eq. (17) that this is a stable mode for $p^2>4q$ and a stable focus for $p^2<4q$. By analogy with Fig. 2 it can be expected that the stochastic realizations converge to the fixed point $\bar{x}_1 = 0$. This presumption is confirmed by Fig. 4 where computations were carried out for $p = 0.09,\ q = 1,\ r = 0.3$. The convergence of the realizations in Fig. 4(c) is not so speedy as in Fig. 2. The fact that this is not a common rule is demonstrated in Fig. 5,

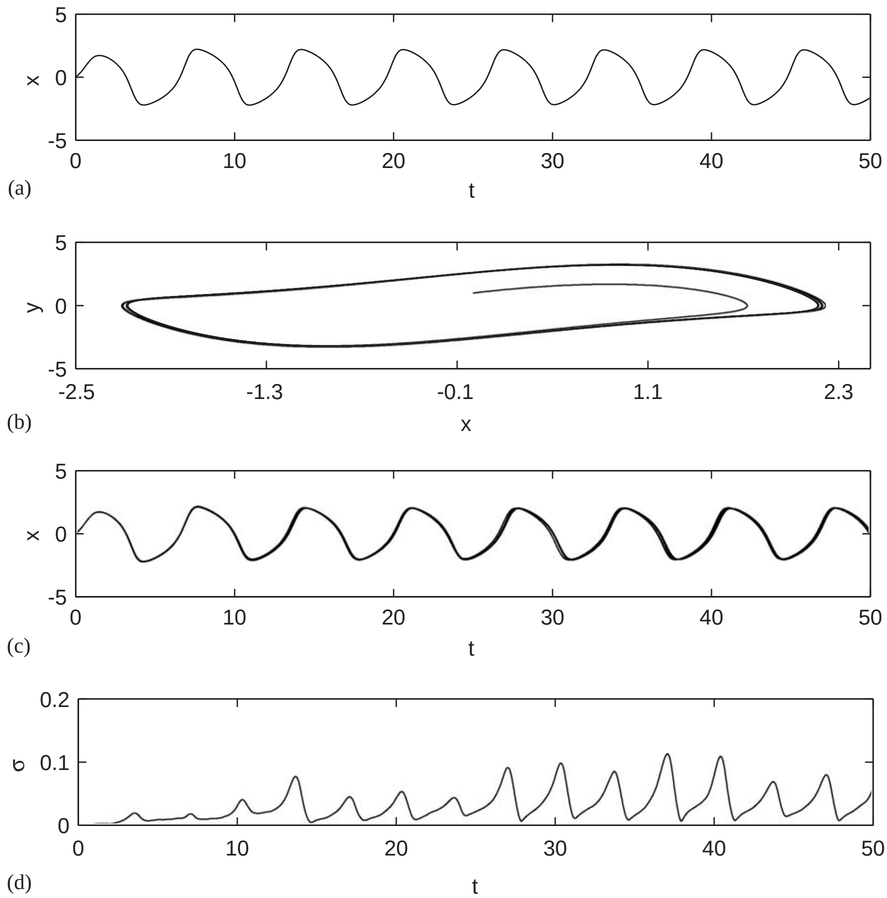


Fig. 7. Van der Pol oscillator (18) for $a = 1,\ q = 1,\ r = 0,\ \omega_0 = 1,\ s = 0.5,\ x_0 = 0,\ \dot{x}_0 = 1$.

where all stochastic realizations practically coincide and the standard deviation is very small ($\sigma < 5 \times 10^{-3}$).

(iii) Assuming $q = 0$ in Eq. (16) the Ueda equation is obtained. This equation has only one fixed point $\bar{x}_1 = 0$; according to Eq. (17) $\lambda_1 = 0$, $\lambda_2 < 0$; consequently this is a degenerated fixed point. Computer simulation results for $p = 0.05,\ q = 0,\ r = 1$ are plotted in Fig. 6. No convergence between different stochastic realizations is observed; the standard deviation $\sigma$ also differs essentially from zero values. In view of the Hartman–Grobman theorem all this was to be expected.

## 6. Van der Pol–Duffing oscillator

The differential equation of this oscillator can be written in the form

$$\ddot{x} - a(1 - x^2)\dot{x} + qx + rx^3 = s \cos \omega t, \quad (a > 0). \tag{18}$$

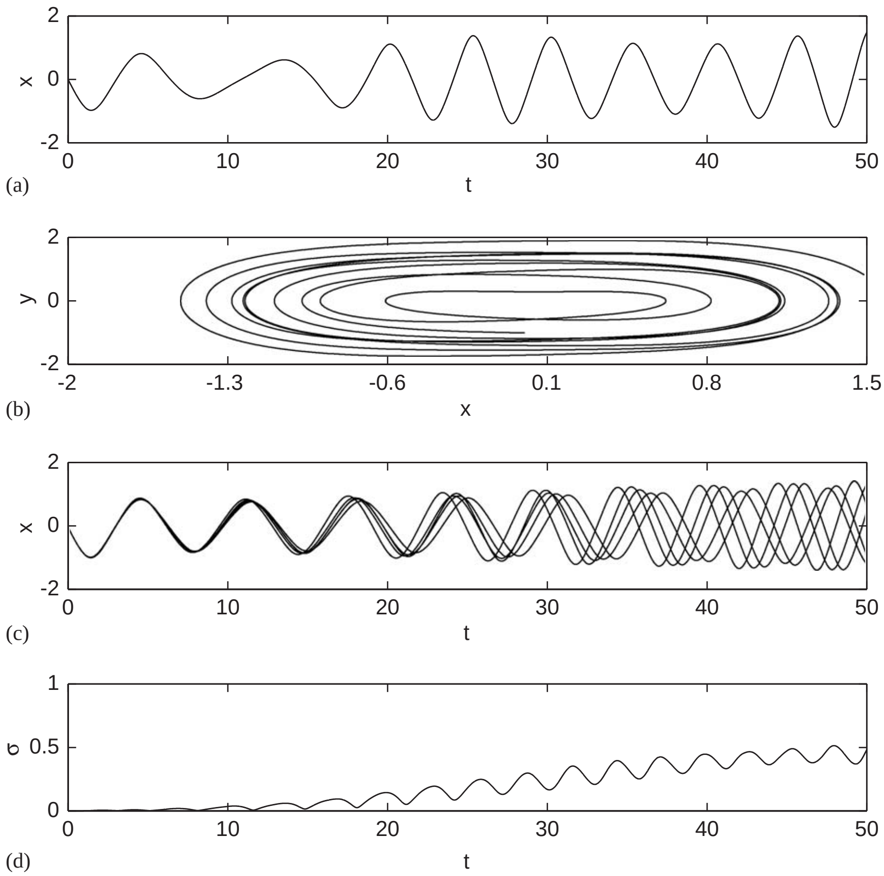


Fig. 8. Van der Pol oscillator (18) for $a = 0.05$, $q = 1$, $r = 1$, $\omega_0 = 0.38$, $s = 0.16$, $x_0 = 0$, $\dot{x}_0 = -1$.

For the conventional van der Pol equation $r = 0$. Since the term $rx^3$ is characteristic to the Duffing equation, then Eq. (18) is called the van der Pol–Duffing equation.

The unforced equation $s = 0$ has only one fixed point $\bar{x} = 0$. Linearization of Eq. (18) in the neighbourhood of the fixed point gives $\ddot{x} - a\dot{x} + qx = 0$. This equation has the eigenvalues

$$\lambda_{1,2} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - q}. \tag{19}$$

If $a^2 > 4q$, then $\lambda_1 > 0$, $\lambda_2 > 0$ and the fixed point is an unstable node; if $a^2 < 4q$, the eigenvalues are complex with a positive real part and the fixed point is unstable focus. Hence it follows that Eq. (18) does not have any stable fixed point. But it is well known that the van der Pol equation may have a limit cycle.

Computer simulation results for $q = 1$, $r = 0$, $s = 0.5$, $\omega = 1$ are plotted in Fig. 7. It can be seen from Fig. 7b that a limit cycle exists. The effect of noise to the vibrations is very small (Fig. 7c): all stochastic realizations practically coincide. A wholly different situation appears for

$a = 0.05,\ q = 1,\ \omega_0 = 0.38,\ r = 1,\ s = 0.16$ (Fig. 8). Here no limit cycle exists; the stochastic realizations diverge and $\sigma$ has a tendency to increase.

## 7. Vibrations of a pendulum

Consider a mathematical pendulum with mass $m$ and length $l$. It is periodically driven by an external force $F = G\cos\,\Omega t$, where $G$ and $\Omega$ are amplitude and frequency of the excitation force. The equation of motion is

$$ml\frac{\mathrm{d}^2\varphi}{\mathrm{d}t_*^2} = -\mu\frac{\mathrm{d}\varphi}{\mathrm{d}t_*} - (mg + G\cos\,\Omega t_*)\sin\,\varphi. \tag{20}$$

Here $\varphi$ is the rotation angle, $g$-gravity constant, $\mu$-damping coefficient.

By the change of variables

$$t_* = t\sqrt{\frac{l}{g}},\quad \omega = \Omega\sqrt{\frac{l}{g}},\quad a = \frac{G}{mg},\quad b = \frac{\mu}{m}\frac{1}{\sqrt{lg}}, \tag{21}$$

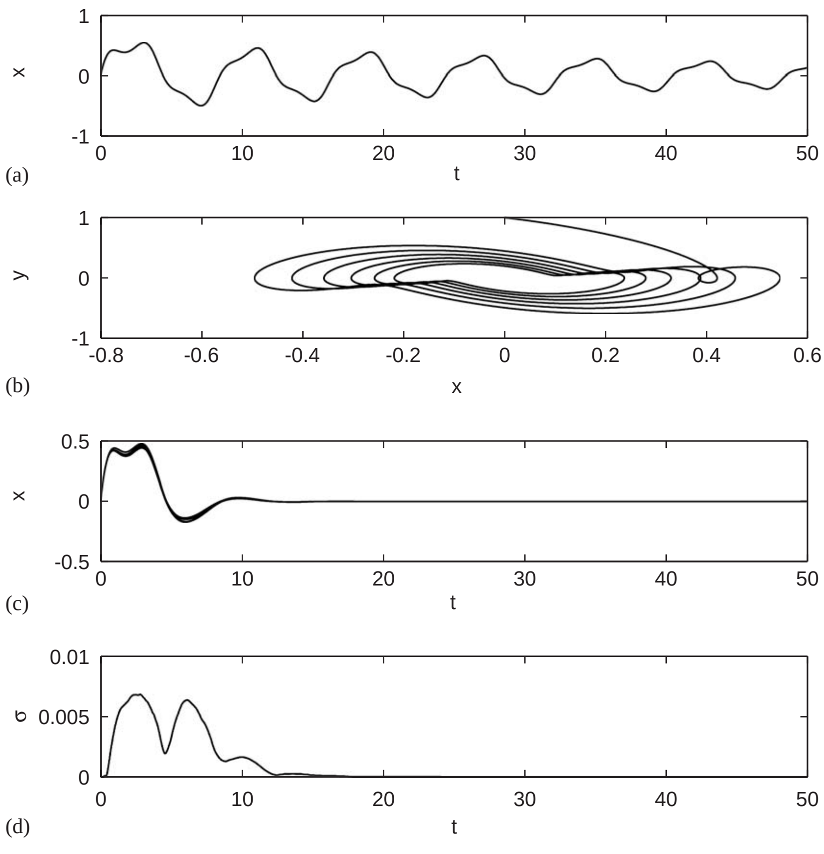


Fig. 9. Driven pendulum (22) for $a = 2,\ b = 1,\ \omega_0 = 0.5\pi,\ x(0) = 0,\ y(0) = 1$.

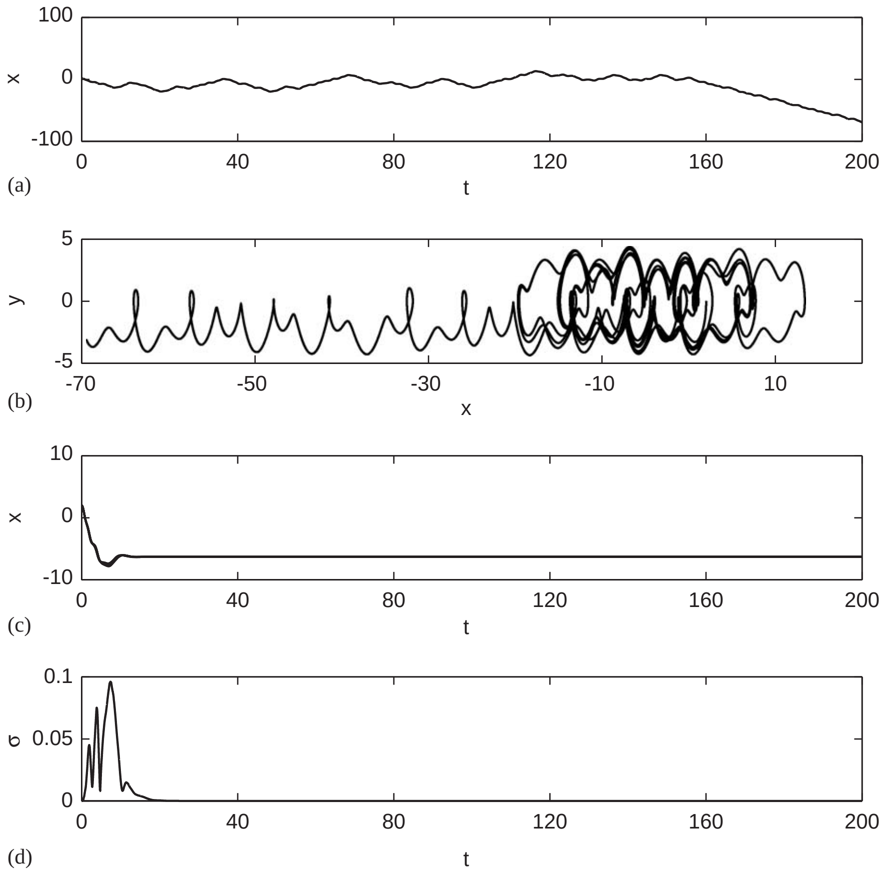


Fig. 10. Driven pendulum (22) for $a = 8,\ b = 1,\ \omega_0 = 0.5\pi,\ x(0) = 2,\ y(0) = 0$.

Eq. (20) can be written in the form

$$\dot{x} = y, \quad \dot{y} = -\sin x(1 + a\cos \omega t) - by. \tag{22}$$

Here $x = \varphi$, dots denote differentiation with respect to nondimensional time $t$.

Fixed points of Eq. (22) are $\bar{x} = k\pi,\ \bar{y} = 0$, where $k$ is an integer. It is shown [23] that if $k$ is an even number the fixed points are stable focuses and saddle points if $k$ is odd.

Depending upon the initial conditions, the motion can be libration, rotation or consist of librations and rotations. As before it is assumed that $\omega$ is stochastic and defined by Eq. (5).

From computer simulations here are presented the results of the following two cases.

(i) The case $a = 2,\ b = 1,\ \omega_0 = 0.5\pi,\ x(0) = 0,\ y(0) = 1$ is plotted in Fig. 9. It follows from Fig. 9a,b that the motion is a nonregular libration. All the stochastic realizations practically coincide and already for $t > 10$ the motion is terminated at the fixed point $\bar{x} = 0$. The standard deviation is very small.

(ii) Here computations were carried out for $a = 8,\ b = 1,\ \omega_0 = 0.5\pi,\ x(0) = 2,\ y(0) = 0$; the results are plotted in Fig. 10. The deterministic motion is irregular, it consists of successive

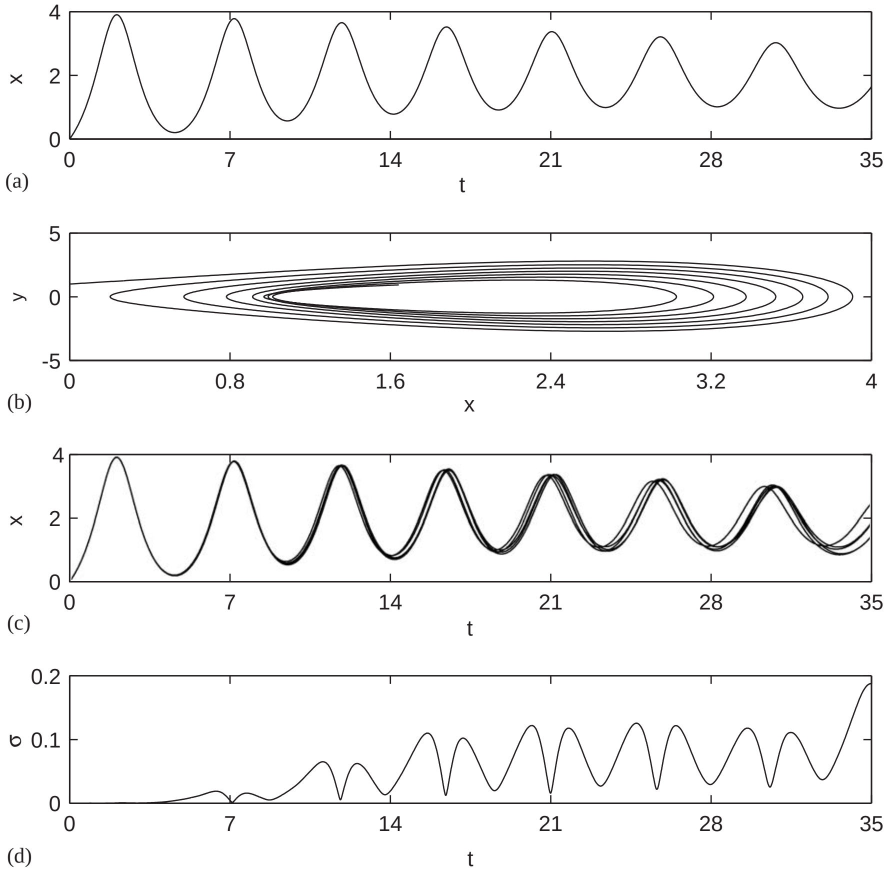


Fig. 11. Duffing equation for $p = 0.05,\ q = -1,\ r = 0.2,\ s = 1,\ \omega_0 = 0.05,\ x_0 = 0,\ \dot{x}_0 = 1.$

librations and rotations. The phase diagram has a rather complicated form. As to noisy motion, then it is very simple: the vibrations die away very soon and the motion terminates in the focus $\bar{x} = -2\pi$.

## 8. Comparison of methods

In the present method, the parameters in system (6), (7) are chosen as follows

$$p = 0.05, q = -1, s = 1, \omega = 0.05, r = 0.2, x_0 = 0, \dot{x}_0 = 1. \tag{23}$$

The numerical results of computer simulation corresponding to Fig. 1 are shown in Fig. 11. The comparison of expectations $E(x)$ calculated by the present method and with the aid of computer simulation for $N = 100$ is presented in Fig. 12. The cases (a)–(c) correspond to the different values of nonlinearity parameter $r = 0.1; 0.5; 1.0$, respectively. Fig. 12 shows good agreement in the case of small values of $r$. The comparison of expectations $E(x)$ in the case of fixed nonlinearity ($r = 0.2$) and different values of noise intensity is shown in Fig. 13. The corresponding lines almost

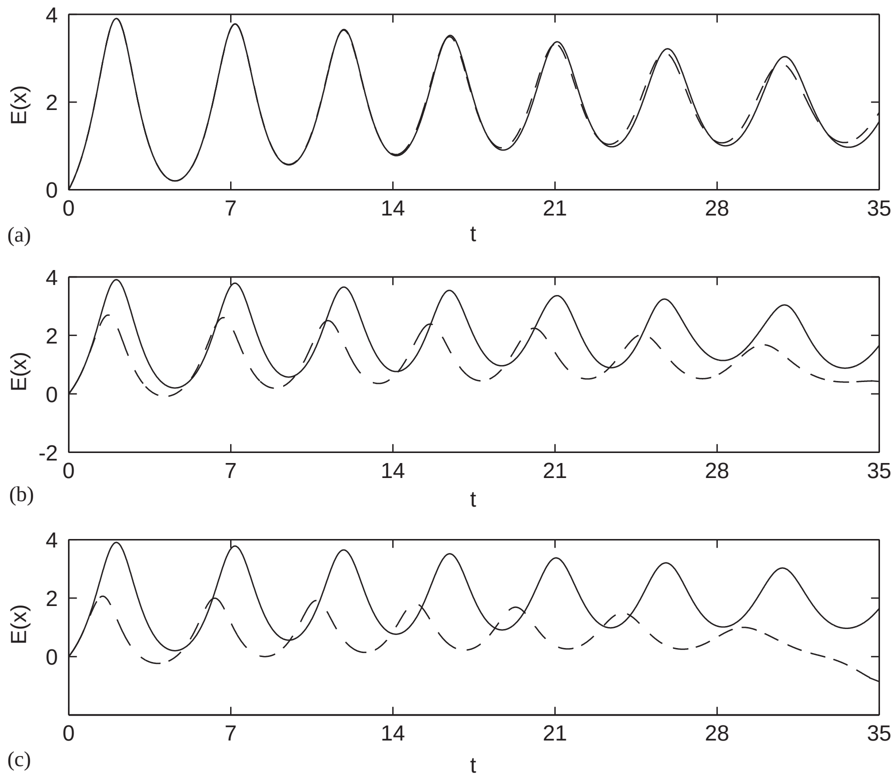


Fig. 12. Comparison of expectations $E(x)$ for $p = 0.05$, $q = -1$, $s = 1$, $\omega_0 = 0.05$, $x_0 = 0$, $\dot{x}_0 = 1$, $\alpha = 0.2$; (a) $r = 0.2$, (b) $r = 0.5$, (c) $r = 1.0$: —— present method, – – – computer simulation (15) where $N = 100$.

coincide. For getting numerical estimates the expectations $E(x)$ were integrated over $t \in [0, T]$ and differences for both methods were calculated. Percentage values of these differences for some values of the parameters $r$ and $\alpha$ are presented in Table 1.

## 9. Conclusions

Nonlinear vibrations with random frequency of excitation are investigated. Two methods of solution are suggested. For weak nonlinearity the equations of motion are linearized. Making use of stochastic averaging, the mean and the variance for the system variable are calculated.

In the case of strong nonlinearity, computer simulation approach is used. By the Runge–Kutta technique stochastic realizations of the system are computed. Divergence of these realizations is estimated by standard deviation. Calculations which were carried out for the Duffing, Ueda, van der Pol attractor and for a periodically driven pendulum, showed that the behaviour of the noisy system essentially depends upon the type of the fixed points. If the fixed points are stable nodes or focuses, then the motion, which for the deterministic system could be chaotic, by adding noise

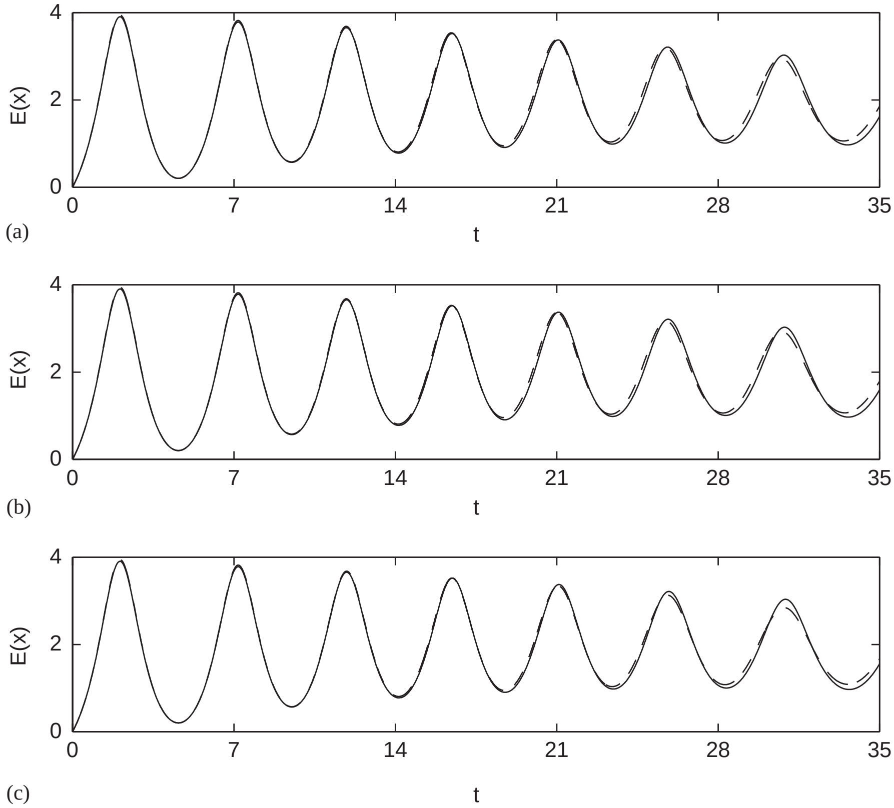


Fig. 13. Comparison of expectations $E(x)$ for $p = 0.05$, $q = -1$, $r = 0.2$, $s = 1$, $\omega_0 = 0.05$, $x_0 = 0$, $\dot{x}_0 = 1$; (a) $\alpha = 0.1$, (b) $\alpha = 0.15$, (c) $\alpha = 0.2$: —— present method, – – – computer simulation (15) where $N = 100$.

Table 1
Percentage values of the differences for integrated expectations in the case of the two methods

| $r$ | $\alpha$ | | |
|---|---|---|---|
| | 0.1 | 0.2 | 0.3 |
| 0.1 | 0.124 | 0.167 | 0.453 |
| 0.3 | 0.588 | 1.208 | 0.889 |
| 0.5 | 1.671 | 2.281 | 2.087 |
| 0.7 | 2.207 | 0.949 | 1.278 |
| 1.0 | 4.683 | 2.189 | 0.614 |

turns regular and terminates in some of the fixed points. If the system has a limit cycle, then the phase portrait of the noisy motion converges to this curve.

In the case of unstable fixed points no convergence of the stochastic realizations is observed.

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## References

[1] J.B. Roberts, P.D. Spanos, *Random Vibration and Statistical Linearization*, Wiley, Chichester, 1990.
[2] P.E. Kloeden, E. Platen, *The Numerical Solution of Stochastic Differential Equations*, Springer, Berlin, 1991.
[3] J.G. McWhirter, I.K. Proudler (Eds.), *Mathematics in Signal Processing*, *V*, Clarendon Press, Oxford, 2002.
[4] P.K. Koliopulos, R.S. Langley, Improved stability analysis of the response of a Duffing oscillator under filtered white noise, *International Journal of Non-Linear Mechanics* 28 (2) (1993) 145–155.
[5] W.Q. Zhu, M.Q. Lu, Q.T. Wu, Stochastic jump and bifurcation of a Duffing oscillator under narrow-band excitation, *Journal of Sound and Vibration* 165 (2) (1993) 285–304.
[6] H. Lin, S.C.S. Yim, Analysis of a nonlinear system exhibiting chaotic, noisy chaotic and random behaviors, *Journal of Applied Mechanics* 63 (1996) 509–516.
[7] S. Benfratello, Pseudo-force method for a stochastic analysis of nonlinear systems, *Probabilistic Engineering Mechanics* 11 (1996) 113–123.
[8] J.G. Wei, G. Leng, Lyapunov exponent and chaos of Duffing's equation perturbed by white noise, *Applied Mathematics and Computation* 88 (1997) 77–93.
[9] H. Rong, W. Xu, T. Fang, Principal response of Duffing oscillator to combined deterministic and narrow-band random parametric excitation, *Journal of Sound and Vibration* 210 (4) (1998) 483–515.
[10] B. Basu, V.K. Gupta, Wavelet-based stochastic seismic response of a Duffing oscillator, *Journal of Sound and Vibration* 245 (2) (2001) 251–260.
[11] W.Y. Liu, W.Q. Zhu, Z.L. Huang, Effect of bounded noise on chaotic motion of Duffing oscillator under parametric excitation, *Chaos Solitons and Fractals* 12 (2001) 527–537.
[12] S. Datta, J.K. Bhattacharjee, Effect of stochastic forcing in the Duffing oscillator, *Physics Letters A* 283 (2001) 323–326.
[13] R. Haiwu, X. Wei, M. Guang, F. Tong, Response of a Duffing oscillator to combined deterministic harmonic and random excitation, *Journal of Sound and Vibration* 242 (2) (2001) 362–368.
[14] C. Proppe, Stochastic linearization of dynamical systems under parametric Poisson white noise excitation, *International Journal of Non-Linear Mechanics* 38 (2003) 543–555.
[15] Z.L. Huang, W.Q. Zhu, Y.Q. Ni, J.M. Ko, Stochastic averaging of strongly non-linear oscillators under bounded noise excitation, *Journal of Sound and Vibration* 254 (2) (2002) 245–267.
[16] W.-C. Xie, Effect of noise on chaotic motion of buckled column under periodic excitation, *Nonlinear and Stochastic Dynamics* 78 (1994) 215–225.
[17] H.K. Leung, Stochastic Hopf bifurcation in a biased van der Pol model, *Physica A* 254 (1998) 146–155.
[18] V. Berdichevsky, M. Gitterman, Stochastic resonance and ratchets—new manifestations, *Physica A* 249 (1998) 88–95.
[19] A.C.J. Luo, R.P.S. Han, The dynamics of stochastic and resonant layers is a periodically driven pendulum, *Solitons and Fractals* 11 (2000) 2349–2359.
[20] W. Szemplińska-Stupnicka, Nonstationary motions in stationary vibrating systems: a discussion, *Dynamics and Vibration of Time-varying Systems and Structures* 56 (1993) 421–425.
[21] Ü. Lepik, Exploring irregular vibrations and chaos by the wavelet method, *Proceedings of the Estonian Academy of Science and Engineering* 9 (2003) 3–24.
[22] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964.
[23] T.S. Parker, L.O. Chua, *Practical Numerical Algorithms for Chaotic Systems*, Springer, Berlin, 1989.